

# ON THE NONTRIVIAL REAL ZERO OF THE REAL PRIMITIVE CHARACTERISTIC DIRICHLET L FUNCTION

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**Abstract.** This paper prove that the real primitive characteristic Dirichlet L function does not exist the non-trivial real zero .

**Keyword.** Dirichlet L function , Real zero

**AMS subject classification.** 11M20

Set  $s = \sigma + it$  is the complex number , suppose that  $Re s > 1$  , the definition of Riemann Zeta function is

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}$$

The definition of Dirichlet L function is

$$L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}$$

where  $\chi(n)$  is Dirichlet characteristic of mod  $q$  .

The product of  $\zeta(s)$  and  $L(s, \chi)$  is

$$\zeta(s)L(s, \chi) = \sum_{n=1}^{+\infty} \frac{a_n}{n^s}$$

where  $a_n = \sum_{d|n} \chi(d)$  . If  $n = p_1^{\alpha_1} \cdots p_u^{\alpha_u}$  is the standard prime factor expression of  $n$  , then from multiplicative of  $a_n$  , have

$$a_n = \prod_{r=1}^u (1 + \chi(p_r) + \cdots + \chi(p_r^{\alpha_r}))$$

easy to see , if  $\chi$  is the real characteristic, then there must be  $a_n \geq 0$  .

**LEMMA 1.** (1) When  $0 < \sigma < 1$  , there is a constant  $0 \leq \alpha \leq 1$  , we have the formula below established

$$\sum_{1 \leq n \leq \xi} \frac{1}{n^\sigma} = \frac{\xi^{1-\sigma} - 1}{1 - \sigma} + \alpha + O\left(\frac{1}{\xi^\sigma}\right)$$

(2) when  $1 < \sigma$  ,  $1 \leq \xi$  , we have

$$\sum_{\xi \leq n} \frac{1}{n^\sigma} = \frac{1}{(\sigma - 1)\xi^{\sigma-1}} + O\left(\frac{1}{\xi^\sigma}\right)$$

The proof of the lemma see the theorem 2 of the page 101 , and the example 4 of the page 103 , of the references [1]

**LEMMA 2.** Set  $s = \sigma + it$  is the complex number , when  $\sigma > 0$  ,  $x \geq 4q(|t| + 2)$  , for the arbitrary nonprincipal character  $\chi$  of mod  $q$  , have the formula below established

$$L(s, \chi) = \sum_{1 \leq n \leq x} \frac{\chi(n)}{n^s} + O(qx^{-\sigma})$$

The proof of the lemma see the theorem 1 of the page 447 of the references [ 2 ]

**LEMMA 3.** Set  $\chi$  is the arbitrary real primitive character of mod  $q$  ,  $q \geq 3$  ,  $L(s, \chi)$  is the corresponding Dirichlet L function.

(1) There exist a positive absolute constant  $c_1$  , such that

$$L(1, \chi) \geq c_1 (\sqrt{q} \log^2 q)^{-1}$$

(2) there exist a positive absolute constant  $c_2$  , the real zero  $\beta$  of the function  $L(s, \chi)$  satisfy

$$\beta \leq 1 - c_2 (\sqrt{q} \log^4 q)^{-1}$$

The proof of the lemma see the theorem 2 of the page 296 , and the theorem 3 of the page 299 of the references [ 2 ].

**LEMMA 4.** Set  $\frac{1}{2} \leq \beta < 1$  is a real zero of the real primitive character Dirichlet  $L(s, \chi)$  function , and  $a_n = \sum_{d|n} \chi(n)$  , when positive integer  $x \geq q^6$  , have  
(1)

$$\sum_{1 \leq n \leq x} \frac{a_n}{n^\beta} = \frac{x^{(1-\beta)}}{1-\beta} L(1, \chi) + O(x^{(1-\beta)-\frac{1}{2}} \log x q^{\frac{3}{2}} \log^4 q)$$

(2)

$$\sum_{n=x+1}^{\infty} \frac{a_n}{n^3} = \frac{L(1, \chi)}{2} (x+1)^{-2} + O(x^{-\frac{5}{2}} q)$$

**Proof.** (1) When positive integer  $x \geq q^6$  , have

$$\begin{aligned} \sum_{1 \leq n \leq x} \frac{a_n}{n^\beta} &= \sum_{1 \leq n \leq x} \frac{1}{n^\beta} \sum_{d|n} \chi(d) = \sum_{1 \leq d \leq x} \frac{\chi(d)}{d^\beta} \sum_{m \leq \frac{x}{d}} \frac{1}{m^\beta} \\ &= \sum_{1 \leq d \leq x^{\frac{1}{2}}} \frac{\chi(d)}{d^\beta} \sum_{m \leq \frac{x}{d}} \frac{1}{m^\beta} + \sum_{x^{\frac{1}{2}} \leq d \leq x} \frac{\chi(d)}{d^\beta} \sum_{m \leq \frac{x}{d}} \frac{1}{m^\beta} = \sum_1 + \sum_2 \end{aligned}$$

From lemma 1 (1) , lemma 2 , and lemma 3 (2) , have

$$\begin{aligned} \sum_1 &= \sum_{1 \leq d \leq x^{\frac{1}{2}}} \frac{\chi(d)}{d^\beta} \left( \frac{x^{1-\beta}}{(1-\beta)d^{1-\beta}} - \frac{1}{1-\beta} + \alpha + O(d^\beta x^{-\beta}) \right) \\ &= \frac{x^{1-\beta}}{1-\beta} \sum_{1 \leq d \leq x^{\frac{1}{2}}} \frac{\chi(d)}{d} + \left( \alpha - \frac{1}{1-\beta} \right) \sum_{1 \leq d \leq x^{\frac{1}{2}}} \frac{\chi(d)}{d^\beta} + O(x^{-\beta+\frac{1}{2}}) \end{aligned}$$

$$\begin{aligned}
&= \frac{x^{1-\beta}}{1-\beta} L(1, \chi) + O\left(\frac{x^{\frac{1}{2}-\beta}}{1-\beta} q\right) + O\left(\frac{x^{-\frac{1}{2}\beta}}{1-\beta} q\right) + O\left(x^{-\beta+\frac{1}{2}}\right) \\
&= \frac{x^{1-\beta}}{1-\beta} L(1, \chi) + O\left(x^{(1-\beta)-\frac{1}{2}} q^{\frac{3}{2}} \log^4 q\right)
\end{aligned}$$

$$\begin{aligned}
\sum_2 &= \sum_{1 \leq m \leq x^{\frac{1}{2}}} \frac{1}{m^\beta} \sum_{x^{\frac{1}{2}} \leq d \leq \frac{x}{m}} \frac{\chi(d)}{d^\beta} = O\left(\sum_{1 \leq m \leq x^{\frac{1}{2}}} \frac{1}{m^\beta} \left(x^{-\frac{1}{2}\beta} \sqrt{q} \log q\right)\right) \\
&= O\left(x^{\frac{1}{2}(1-\beta)} x^{-\frac{1}{2}\beta} \log x \sqrt{q} \log q\right) = O\left(x^{(1-\beta)-\frac{1}{2}} \log x \sqrt{q} \log q\right)
\end{aligned}$$

(2) Set  $y = x + 1$ , and the positive integer  $x \geq q^6$ , then

$$\begin{aligned}
\sum_{y \leq n < \infty} \frac{a_n}{n^3} &= \sum_{y \leq n < \infty} \frac{1}{n^3} \sum_{d|n} \chi(d) = \sum_{1 \leq d < \infty} \frac{\chi(d)}{d^3} \sum_{\frac{y}{d} \leq m < \infty} \frac{1}{m^3} \\
&= \sum_{1 \leq d < y^{\frac{1}{2}}} \frac{\chi(d)}{d^3} \sum_{\frac{y}{d} \leq m < \infty} \frac{1}{m^3} + \sum_{y^{\frac{1}{2}} \leq d < \infty} \frac{\chi(d)}{d^3} \sum_{\frac{y}{d} \leq m < \infty} \frac{1}{m^3} = \sum_3 + \sum_4
\end{aligned}$$

From lemma 1 (2) and lemma 2, have

$$\sum_3 = \sum_{1 \leq d \leq y^{\frac{1}{2}}} \frac{\chi(d)}{d^3} \left( \frac{d^2}{2y^2} + O\left(\frac{d^3}{y^3}\right) \right) = \frac{1}{2y^2} \sum_{1 \leq d \leq y^{\frac{1}{2}}} \frac{\chi(d)}{d} + O\left(y^{-\frac{5}{2}}\right)$$

$$= \frac{L(1, \chi)}{2y^2} + O\left(y^{-\frac{5}{2}}q\right)$$

$$\begin{aligned} \sum_4 &= \sum_{1 \leq m \leq y^{\frac{1}{2}}} \frac{1}{m^3} \sum_{\frac{y}{m} \leq d < \infty} \frac{\chi(d)}{d^3} = O\left(\sum_{1 \leq m \leq y^{\frac{1}{2}}} \frac{1}{m^3} \left(\frac{m^3 \sqrt{q} \log q}{y^3}\right)\right) \\ &= O\left(y^{-\frac{5}{2}} \sqrt{q} \log q\right) \end{aligned}$$

This completes the proof.

**LEMMA 5.** Set  $\Gamma(s)$  is Euler  $\Gamma$  function.

(1) When  $\operatorname{Re} s > 0$ , have

$$\Gamma(s) = \int_0^\infty e^{-u} u^{s-1} du$$

(2) When  $y > 0$ ,  $b > 0$ , have

$$e^{-y} = \frac{1}{2\pi i} \int_{(b)} y^{-s} \Gamma(s) ds$$

where  $\int_{(b)} = \int_{b-i\infty}^{b+i\infty}$

(3) Set  $s$  is the arbitrary complex number, we have

$$-\frac{\Gamma'(s)}{\Gamma(s)} = \frac{1}{s} + \gamma + \sum_{n=1}^{\infty} \left( \frac{1}{n+s} - \frac{1}{n} \right)$$

where  $\gamma$  is Euler constant.

The proof of the lemma see the page 20, the properties 4 of the page 45, and the properties 8 of the page 48, of the references [2]

**LEMMA 6.** Set  $\Gamma(s)$  is Euler  $\Gamma$  function.

(1) When  $\frac{1}{8} \leq \sigma \leq \frac{1}{2}$ , have  $\Gamma'(\sigma) < 0$ , in other words,  $\Gamma(\sigma)$  decrease monotonically in this interval.

$$(2) \quad \Gamma\left(\frac{1}{3}\right) \geq 2.67, \quad \Gamma\left(\frac{1}{6}\right) \leq 6.$$

**Proof.** (1) When  $\frac{1}{8} \leq \sigma \leq \frac{1}{2}$ , from the lemma 5 (3), have

$$\begin{aligned} -\frac{\Gamma'(\sigma)}{\Gamma(\sigma)} &= \frac{1}{\sigma} + \gamma + \sum_{n=1}^{\infty} \left( \frac{1}{n+\sigma} - \frac{1}{n} \right) \\ \frac{\Gamma'(\sigma)}{\Gamma(\sigma)} &= -\frac{1}{\sigma} - \gamma + \sigma \sum_{n=1}^{\infty} \frac{1}{(n+\sigma)n} \\ &\leq -2 - \gamma + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} = -2 - \gamma + \frac{\pi^2}{12} < -1 \end{aligned}$$

Because when  $\frac{1}{8} \leq \sigma \leq \frac{1}{2}$ , have  $\Gamma(\sigma) > 0$ , so  $\Gamma'(\sigma) < 0$ , namely,  $\Gamma(\sigma)$  decrease monotonically in this interval.

(2) From the current lemma (1), have  $\Gamma\left(\frac{1}{3}\right) \geq \Gamma(0.334)$ , from the functional equation  $\Gamma(s+1) = s\Gamma(s)$ , and consult the page 1312 of the references [3], have  $\Gamma(0.334) = \frac{\Gamma(1.334)}{0.334} \geq \frac{0.8929}{0.334} \geq 2.67$ . On the same score,  $\Gamma\left(\frac{1}{6}\right) \leq \Gamma(0.166) = \frac{\Gamma(1.166)}{0.166} \leq \frac{0.928}{0.166} \leq 6$ . This completes the proof.

**LEMMA 7.** Set  $\chi$  is a real primitive characteristic of mod  $q$ , and  $a_n = \sum_{d|n} \chi(d)$ . When  $y > 0$ ,  $\operatorname{Re} s > \frac{1}{3}$ , we have

$$\begin{aligned} \int_0^y \left( \sum_{n=1}^{+\infty} a_n e^{-n^3 u} \right) u^{s-1} du &= L(1, \chi) \Gamma\left(\frac{1}{3}\right) \frac{3y^{s-\frac{1}{3}}}{3s-1} + \delta L(0, \chi) \zeta(0) \frac{y^s}{s} \\ &+ \int_{(-\frac{1}{3})} L(3w, \chi) \zeta(3w) \Gamma(w) \frac{y^{s-w}}{s-w} dw \end{aligned}$$

where  $\delta = \frac{1}{2}(1 - \chi(-1))$

**Proof.** From the lemma 5 (2), when  $u > 0$ , we have

$$e^{-u} = \frac{1}{2\pi i} \int_{(2)} u^{-w} \Gamma(w) dw$$

Set  $n$  is a positive integer , now let us do the integral transformation in the above formula :  $u \rightarrow n^3 u$  , we have

$$\begin{aligned} e^{-n^3 u} &= \frac{1}{2\pi i} \int_{(2)} \frac{u^{-w}}{n^{3w}} \Gamma(w) dw \\ \sum_{n=1}^{+\infty} a_n e^{-n^3 u} &= \frac{1}{2\pi i} \int_{(2)} \left( \sum_{n=1}^{+\infty} \frac{a_n}{n^{3w}} \right) u^{-w} \Gamma(w) dw \\ &= \frac{1}{2\pi i} \int_{(2)} L(3w, \chi) \zeta(3w) u^{-w} \Gamma(w) dw \end{aligned}$$

When  $Re w \geq -\frac{1}{3}$  , integrand function in  $w = \frac{1}{3}$  place has a pole of order 1. When  $\delta = 1$  , integrand function in  $w = 0$  place has a pole of order 1. from the residue theorem have the above formula

$$= L(1, \chi) \Gamma\left(\frac{1}{3}\right) u^{-\frac{1}{3}} + \delta L(0, \chi) \zeta(0) + \frac{1}{2\pi i} \int_{(-\frac{1}{3})} L(3w, \chi) \zeta(3w) \Gamma(w) u^{-w} dw$$

Suppose that  $Re s > \frac{1}{3}$  ,  $y > 0$  the above equation multiply by  $u^{s-1}$  on both sides , then integral , we have

$$\begin{aligned} \int_0^y \left( \sum_{n=1}^{+\infty} a_n e^{-n^3 u} \right) u^{s-1} du &= L(1, \chi) \Gamma\left(\frac{1}{3}\right) \int_0^y u^{s-\frac{4}{3}} du \\ &+ \delta L(0, \chi) \zeta(0) \int_0^y u^{s-1} du + \frac{1}{2\pi i} \int_{(-\frac{1}{3})} L(3w, \chi) \zeta(3w) \Gamma(w) \left( \int_0^y u^{-w+s-1} du \right) dw \\ &= L(1, \chi) \Gamma\left(\frac{1}{3}\right) \frac{3 y^{s-\frac{1}{3}}}{3s-1} + \delta L(0, \chi) \zeta(0) \frac{y^s}{s} + \frac{1}{2\pi i} \int_{(-\frac{1}{3})} L(3w, \chi) \zeta(3w) \Gamma(w) \frac{y^{s-w}}{s-w} dw \end{aligned}$$

This completes the proof.

**THEOREM.** Set  $\chi$  is a real primitive characteristic of mod  $q$  , the corresponding Dirichlet L function  $L(s, \chi)$  does not exist the nontrivial real zero.

**Proof.** Set  $n$  is a positive integer and  $Re s > \frac{1}{3}$  , as variable transformation for below formula :  $n^3 u \rightarrow u$  , from the lemma 5 (1) , have

$$\int_0^\infty e^{-n^3 u} u^{s-1} du = \frac{1}{n^{3s}} \int_0^\infty e^{-u} u^{s-1} du = \frac{1}{n^{3s}} \Gamma(s)$$

so

$$\begin{aligned} \Gamma(s) \sum_{n=1}^\infty \frac{a_n}{n^{3s}} &= \int_0^\infty \left( \sum_{n=1}^{+\infty} a_n e^{-n^3 u} \right) u^{s-1} du \\ \Gamma(s) L(3s, \chi) \zeta(3s) &= \int_0^\infty \left( \sum_{n=1}^{+\infty} a_n e^{-n^3 u} \right) u^{s-1} du \\ &= \int_y^\infty \left( \sum_{n=1}^{+\infty} a_n e^{-n^3 u} \right) u^{s-1} du + \int_0^y \left( \sum_{n=1}^{+\infty} a_n e^{-n^3 u} \right) u^{s-1} du \end{aligned}$$

where  $y > 0$ .

from the lemma 7, we have

$$\begin{aligned} \Gamma(s) L(3s, \chi) \zeta(3s) &= \int_y^\infty \left( \sum_{n=1}^{+\infty} a_n e^{-n^3 u} \right) u^{s-1} du \\ &+ L(1, \chi) \Gamma\left(\frac{1}{3}\right) \frac{3 y^{s-\frac{1}{3}}}{3s-1} + \delta L(0, \chi) \zeta(0) \frac{y^s}{s} + \frac{1}{2\pi i} \int_{(-\frac{1}{3})} L(3w, \chi) \zeta(3w) \Gamma(w) \frac{y^{s-w}}{s-w} dw \end{aligned}$$

From the above formula, We put the function  $\Gamma(s) L(3s, \chi) \zeta(3s)$  analytic continuation to the half plane  $\operatorname{Re} s > -\frac{1}{3}$ , this function have a pole of order 1 on  $s = \frac{1}{3}$ , and when  $\delta = 1$ , have a pole of order 1 on  $s = 0$ .

Now assume,  $L(s, \chi)$  exists a nontrivial real zero  $\beta$ , from the functional equation of  $L(s, \chi)$ , we can assume that  $\frac{1}{2} \leq \beta < 1$ .

Now take  $s = \frac{\beta}{3}$ , we have

$$\begin{aligned} 0 &= \int_y^\infty \left( \sum_{n=1}^{+\infty} a_n e^{-n^3 u} \right) u^{\frac{\beta}{3}-1} du + L(1, \chi) \Gamma\left(\frac{1}{3}\right) \frac{3 y^{\frac{\beta-1}{3}}}{\beta-1} \\ &+ \delta L(0, \chi) \zeta(0) \frac{3 y^{\frac{\beta}{3}}}{\beta} + \frac{1}{2\pi i} \int_{(-\frac{1}{3})} L(3w, \chi) \zeta(3w) \Gamma(w) \frac{3 y^{\frac{\beta}{3}-w}}{\beta-3w} dw \end{aligned}$$



in other words

$$L(1, \chi) \Gamma\left(\frac{1}{3}\right) \frac{3y^{\frac{\beta-1}{3}}}{1-\beta} = \int_y^\infty \left( \sum_{n=1}^{+\infty} a_n e^{-n^3 u} \right) u^{\frac{\beta}{3}-1} du$$

$$+ \delta L(0, \chi) \zeta(0) \frac{3y^{\frac{\beta}{3}}}{\beta} + \frac{1}{2\pi i} \int_{(-\frac{1}{3})} L(3w, \chi) \zeta(3w) \Gamma(w) \frac{3y^{\frac{\beta}{3}-w}}{\beta-3w} dw = I_1 + I_2 + I_3$$

Now take  $y = x^{-3}$ ,  $x$  is a positive integer, and  $x \geq q^6$ . we calculated the above formula every one on the right

$$I_2 = O\left(\delta L(1, \chi) q^{\frac{1}{2}} y^{\frac{1}{6}}\right) = O\left(x^{-\frac{1}{2}} q^{\frac{1}{2}} \log q\right)$$

From the functional equation of  $L(s, \chi)$ ,  $\zeta(s)$ , and the asymptotic formula of  $\Gamma(s)$ , we have

$$|I_3| \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} |L(-1+i\nu, \chi)| |\zeta(-1+i\nu)| \left| \Gamma\left(-\frac{1}{3}+i\nu\right) \right| \left| \frac{3y^{\frac{1+\beta}{3}}}{\beta+1-3i\nu} \right| d\nu$$

$$= O\left(q^{\frac{3}{2}} y^{\frac{1+\beta}{3}}\right) = O\left(x^{-\frac{3}{2}} q^{\frac{3}{2}}\right)$$

Now calculate  $I_1$

$$I_1 = \sum_{n=1}^x a_n \int_y^\infty e^{-n^3 u} u^{\frac{\beta}{3}-1} du + \sum_{n=x+1}^{+\infty} a_n \int_y^\infty e^{-n^3 u} u^{\frac{\beta}{3}-1} du = J_1 + J_2$$

Now calculate  $J_1$

$$J_1 \leq \sum_{n=1}^x a_n \int_0^\infty e^{-n^3 u} u^{\frac{\beta}{3}-1} du$$

as variable transformation for above formula :  $n^3 u \rightarrow u$ , have the above formula

$$= \sum_{n=1}^x \frac{a_n}{n^\beta} \int_0^\infty e^{-u} u^{\frac{\beta}{3}-1} du = \Gamma\left(\frac{\beta}{3}\right) \sum_{n=1}^x \frac{a_n}{n^\beta}$$

From the lemma 4 (1) , have the above formula

$$= \Gamma\left(\frac{\beta}{3}\right) \frac{x^{(1-\beta)}}{1-\beta} L(1, \chi) + O(x^{(1-\beta)-\frac{1}{2}} q^{\frac{3}{2}} \log x \log^4 q)$$

From the lemma 6 (1) and (2), have the above formula

$$\begin{aligned} &\leq \Gamma\left(\frac{1}{6}\right) \frac{x^{(1-\beta)}}{1-\beta} L(1, \chi) + O(x^{(1-\beta)-\frac{1}{2}} q^{\frac{3}{2}} \log x \log^4 q) \\ &\leq 6 \frac{x^{(1-\beta)}}{1-\beta} L(1, \chi) + O(x^{(1-\beta)-\frac{1}{2}} q^{\frac{3}{2}} \log x \log^4 q) \end{aligned}$$

Now calculate  $J_2$

$$J_2 \leq y^{\frac{\beta}{3}-1} \sum_{n=x+1}^{+\infty} a_n \int_y^\infty e^{-n^3 u} du$$

as variable transformation for above formula :  $n^3 u \rightarrow u$  , have the above formula

$$= y^{\frac{\beta}{3}-1} \sum_{n=x+1}^{+\infty} \frac{a_n}{n^3} \int_{n^3 y}^\infty e^{-u} du \leq y^{\frac{\beta}{3}-1} \sum_{n=x+1}^{+\infty} \frac{a_n}{n^3} \int_1^\infty e^{-u} du = \frac{y^{\frac{\beta}{3}-1}}{e} \sum_{n=x+1}^{+\infty} \frac{a_n}{n^3}$$

From the lemma 4 (2) , have the above formula

$$= \frac{y^{\frac{\beta}{3}-1}}{e} \frac{L(1, \chi)}{2} (x+1)^{-2} + O(y^{\frac{\beta}{3}-1} x^{-\frac{5}{2}} q) \leq \frac{x^{(1-\beta)}}{2e} L(1, \chi) + O(x^{(1-\beta)-\frac{1}{2}} q)$$

we synthesize the above calculation , have

$$L(1, \chi) \Gamma\left(\frac{1}{3}\right) \frac{3x^{(1-\beta)}}{1-\beta} \leq 6 \frac{x^{(1-\beta)}}{1-\beta} L(1, \chi) + \frac{x^{(1-\beta)}}{2e} L(1, \chi) + O\left(x^{(1-\beta)-\frac{1}{2}} q^{\frac{3}{2}} \log x \log^4 q\right)$$

From the lemma 6 (2) , have

$$8 L(1, \chi) \frac{x^{(1-\beta)}}{1-\beta} \leq 6 \frac{x^{(1-\beta)}}{1-\beta} L(1, \chi) + \frac{x^{(1-\beta)}}{2e} L(1, \chi) + O\left(x^{(1-\beta)-\frac{1}{2}} q^{\frac{3}{2}} \log x \log^4 q\right)$$

so

$$2 L(1, \chi) \frac{x^{(1-\beta)}}{1-\beta} \leq \frac{x^{(1-\beta)}}{2e} L(1, \chi) + O\left(x^{(1-\beta)-\frac{1}{2}} q^{\frac{3}{2}} \log x \log^4 q\right)$$

Divided by  $2 L(1, \chi) x^{(1-\beta)}$  on both sides , and from the lemma 3 (1) , have

$$\frac{1}{1-\beta} \leq \frac{1}{4e} + O\left(x^{-\frac{1}{2}} q^2 \log x \log^6 q\right)$$

Make  $x \rightarrow +\infty$  , we have

$$\frac{1}{1-\beta} \leq \frac{1}{4e} \leq \frac{1}{10}$$

so  $\beta \leq -9$  , with  $\frac{1}{2} \leq \beta < 1$  contradictions .  
This completes the proof.

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